AIAA JOURNAL
Vol. 47, No. 2, February 2009

Control of Beams and Chains Through Distributed Gyroscopes

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DOI: 10.2514/1.29250

The control of deformable bodies through a large number of small gyroscopes has received attention in the past 20 years. Promising in principle for the control of spacecraft structures, the idea needs the design of fit control algorithms to be developed in practice. Aiming at this design, here we develop two special but technically important cases, which can be easily implemented and of possible use in spacecraft appendages: an elastic beam with an embedded array of coaxial rotors distributed along the beam's axis and spinning about it, and a chain, each rigid link of which contains a rotor whose axis is pinned along the link-to-link direction. We show some results for the dynamics of such structures that are made possible by the control of the rotors and can be implemented to deal with and eventually damp small disturbances: the propagation of screw waves of small amplitude through chains or beams, and the diversion of energy from one mode to another in the linear transversal vibrations of beams.

Nomenclature

$\mathcal{A}_*(\xi_*)$	=	cross section of the chain in the reference configuration
$\mathbf{a}_*$, $\hat{\mathbf{a}}$, $\mathbf{a}$	=	reference vector normal to the gyroscopic axis in the reference configuration, in the actual configuration, and in the same after control
a_2, a_3, b_2, b_3	=	amplitudes of waves
$\mathbf{b}$	=	external force per unit mass in the actual configuration
$\tilde{\mathbf{b}}$	=	external force per unit mass in the actual configuration of the chain
$\mathbf{b}_{Ig}$	=	inertia torque per unit mass due to the gyroscope in the actual configuration
$\bar{c}_\pm$, $\bar{\bar{c}}_\pm$, $\bar{c}_2$	=	constants
E	=	Young's modulus
$\mathbb{E}$	=	Ricci's alternating third-order tensor
$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$	=	reference axes
$\mathbf{F}$	=	deformation gradient
$\mathbf{G}$	=	relative rotation between the gyroscope and the capsule bearing it
$\mathbf{g}_*$, $\hat{\mathbf{g}}$, $\mathbf{g}$	=	gyroscopic axis in the reference configuration, in the actual configuration, and in the same after control
$\mathbb{H}$	=	third-order tensor
$\mathtt{i}$	=	imaginary unit
$\mathbf{J}$	=	inertia tensor of the gyroscope per unit mass in the actual configuration
$\mathbf{L}$	=	velocity gradient
n, m, k	=	(indices) integer numbers
$\mathbf{n}$	=	tension in the chain
$\mathbf{P}$	=	Piola–Kirchooff stress tensor
$\mathbf{p}$	=	gyroscopic precession direction in the actual configuration
$\mathbf{Q}$	=	total rotation of the gyroscope
$\hat{\mathbf{Q}}$	=	entrainment rotation of the capsule bearing the gyroscope
$\bar{q}_2$, $\bar{q}_3$	=	harmonic loading components in the x_2 and x_3 direction on a beam
$\mathcal{S}_*$, $\mathcal{S}$	=	reference and actual configuration of the mean fiber of the chain
$\bar{s}_\pm$, $\bar{\bar{s}}_\pm$, $\bar{s}_2$	=	constants
$\mathbf{T}$	=	Cauchy stress tensor
$\mathbf{u}$	=	displacement of the beam
$\mathbf{w}$, $\mathbf{w}^r$, $\mathbf{w}^e$	=	total, relative, and entrainment spin of gyroscopes
x_*, x	=	reference and actual placement
$\bar{x}$	=	length of the beam
$\tilde{x}$	=	actual placement of the center of mass of the chain's cross section
α_1, α_2	=	eigenvalues of $\mathbf{J}$ (α_1 becomes α for shortness)
β, β_I	=	axial component of the external torque and of the inertial torque due to the gyroscope per unit mass in the actual configuration
γ	=	acceleration of gravity
γ_0	=	weight suspended at the end of the chain per unit reference mass of the chain
ε	=	energy ratio after control
ζ	=	axial component of the equilibrated microtorque
η	=	viscous resistance to relative axial rotation between gyroscope and capsule
μ	=	relative precession speed of the gyroscope
ξ_*, ξ	=	reference and actual curvilinear coordinate along the mean fiber of the chain
ρ_*, ρ	=	mass density in the reference and actual configuration
$\tilde{\rho}_*$	=	mass density of the chain in the reference configuration
ϱ_2, ϱ_3	=	radii of inertia of the beam's cross section
σ	=	real frequency (also with indices jn, $2n$, $3n$, $n\pm$, or $\pm$)
ς	=	complex frequency (also with indices 2 and 3)
τ	=	time (also with overbars to denote given instants)
φ_2, φ_3	=	functions of time

Received 11 December 2006; revision received 24 April 2008; accepted for publication 4 August 2008.

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ω, ω^e, ω_0	=	relative and entrainment angular speed of the gyroscope, a constant relative angular speed
$1/\lambda$	=	complex wavelength

I. Introduction

THE mechanical response of some structures can be controlled through appropriate insertion of sensors, processors, and actuators. Sometimes the transducers can be densely distributed through the structure and used to apply internal or inertial actions. Problems must then be tackled of the type considered in the theory of distributed controls, but the type of mechanical influence used by the control devices becomes paramount for the design of control algorithms, especially because it affects the mathematical nature of the problem. In this paper, we focus on the possible control of structures through continuous distributions of gyroscopes.

In [1], the idea that some *distributed gyricity* can be used to control a flexible structure was introduced. The authors considered a linear elastic body undergoing infinitesimal transformations (small deformations, small translations, and small rotations), in which each volume element possesses an infinitesimal quantity of *stored* angular momentum they called gyricity and conceived as the effect of a very small rotor (actually, an infinitesimal one) embedded into the element. Ideally, material elements are gyroscopes, whose angular momentum is not given by the overall rotation of the body (actually, even not influenced by it), but imposed by some external source: their mass enters the equation of motion through the translational inertia term, whereas their gyricity appears in the same equation as an equivalent external force.

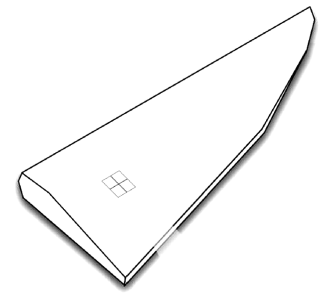
Advantages may arise in control practice, with respect to the use of one or a few gyroscopes, as distributed actuators may better fit with the needs of deformable structures [2–4].

Because of the assumptions, the early theory quoted previously does not take into account the whole coupling between the motion of the material elements (as small parts of the body) and that of the rotors within them. The former do affect, in principle, the angular momentum of each rotor, which should then not be considered as an external supply. Knowledge of this coupling can be decisive to design control strategies based on the guidance of the rotor's spin. Therefore, the theory needs to be extended to finite transformations.

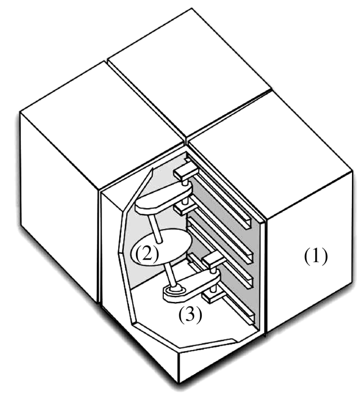
Furthermore, the assumption of linear elasticity is not essential to the theory and perhaps restrictive. Consider, for instance, the case of a tether one wishes to control, modeled as an inextensible perfectly flexible string or chain: though there is no elastic energy in the system, a linear problem can be written to describe it in some circumstances. Thence, a general theory of *gyrocontinua* should be proposed, removing some of the restrictions that have permitted the idea of *gyricity* to appear first.

Following the early proposal and the results and developments that appeared later in the literature, we have considered in [5] a fabric bearing a large number of very small gyroscopes spinning about axes that are pin-fixed (possibly through pliant, controllable links) to a flexible frame (see Fig. 1). Thus, our problem mimics that presented in [1,6,7], but for the fact that we do not need to consider elastic structures only. Therefore, to model this system at a scale fit for engineering applications, we have called upon the theory of continua with microstructure and considered the detailed description of the linkage, in each element, between gyroscope and frame that is given in the first part of this paper. The resulting theoretical model can then find concrete applications, also when the structure behaves (gyroscopic coupling apart) nonlinearly: structures undergoing large rotations or displacements or finite deformations, nonlinear elasticity, geometrically nonlinear systems like ropes, cables, cable nets, or membranes.

Any frame bearing a gyroscope moves under the influence of the latter via the reactions arising in the pins; therefore, the whole structure can be controlled through either the angular speed of the gyroscopes or the inclination of their angular velocity relative to the frame, or in both ways. If the axes of rotation of the gyroscopes are kept fixed along given material fibers, we have a control through the angular speed that, if the latter is also kept fixed, replicates the strategy already pointed out in the quoted literature.



a)



b)

Fig. 1 A gyrocontinuum is a) a body, the material elements of which have b) the structure of gyroscopes: 1) capsule, 2) rotor, and 3) gimbals.

Notice that, in the case when the gyroscopic axes are fixed material fibers, our model gives a local dependence of the Cauchy stress from the angular velocity of the gyroscopes. The traction on material surfaces depends then not only on the particular material behavior (the stress–strain relation in elasticity for instance), but also on this angular velocity, which can be used to affect, and possibly control, the deformation process. Then, problems can be tackled of the type considered in the theory of distributed controls, with this angular velocity as a control variable, but a basic issue is to be pointed out: here, the controls may influence the terms under partial derivative in the equations of motion rather than those expressing external actions (i.e., influence the effective stress rather than the external forces), thus possibly changing the kind of solution of the system [8]. It must be noticed, in fact, that most of the control devices allow one to apply fit external forces on the structure, whereas, here, we are considering a system acting, through the gyroscopes, on the apparent material behavior of the structural elements.

In [5], developing the general theory, we gave hints on some possible concrete applications. Here, we focus on two special but technically important cases, which can be easily implemented experimentally: 1) an elastic beam with an embedded array of coaxial

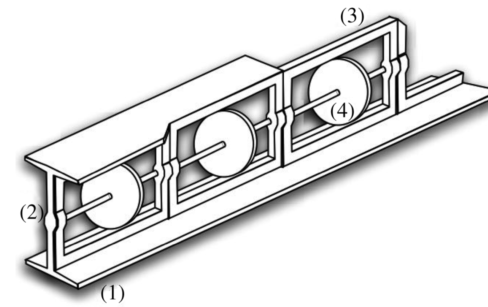


Fig. 2 Sketch of a beam bearing gyroscopes: 1) wing, 2) web equipped with 3) capsules housing 4) rotors.

gyroscopes distributed along the beam's axis and spinning about it (see Fig. 2), and 2) a chain, each rigid link of which contains a gyroscope pinned along the link-to-link direction (see Fig. 3). In both cases, it is easy to imagine rotors spinning with fixed angular speed and a control being performed switching on/off this velocity or switching $+/-$ its sign.

Considering the angular speed as a control field, we can look for appropriate choices of it to achieve a given objective. Choices will possibly be optimal and will eventually occur within limits (for instance, the range of possible angular speeds will be limited). To simplify matters, we will assume the controller is acting instantaneously on each gyroscope, with negligible retroaction, and we will also neglect the minimal time which would technically be necessary to change the value of the angular speed from one extremum of its interval to the other: here, this speed can be switched instantaneously between any two values in the interval where it belongs. When such a sudden change occurs, a perturbation propagates through the material. To simplify matters, it is also assumed that such transitions have negligible effects and they will not be described. Even within these simplifying assumptions, some interesting results can be put forward and some control strategies can be proposed.

II. Gyrocontinua

A gyrocontinuum is the model of a body endowed with the fine structure of very many, very small gyroscopes (see Fig. 1). We thus have to refer to two kinds of motion: the macroscopic and the microscopic one.

The macroscopic motion is the one observed at the scale of the whole structure, not taking into account microscopic details. At this scale, the geometry and kinematics of material elements are seen as those of the points of a standard continuum.

The quoted details are nonetheless of import for the description of the motion of each gyroscope (actually a microscopic device) and thus for the description of the torques of inertia applied by these devices to the structure that eventually influence the deformation process. To handle them, we introduce what we call the microstructure: a set of geometric information that is sufficient to the aforementioned descriptions. To be more precise, this information will actually be represented by an element of a fit manifold of microstructures. The microscopic motion will then be that allowed for the microstructures, that is, an element of the tangent space to the manifold of microstructures.

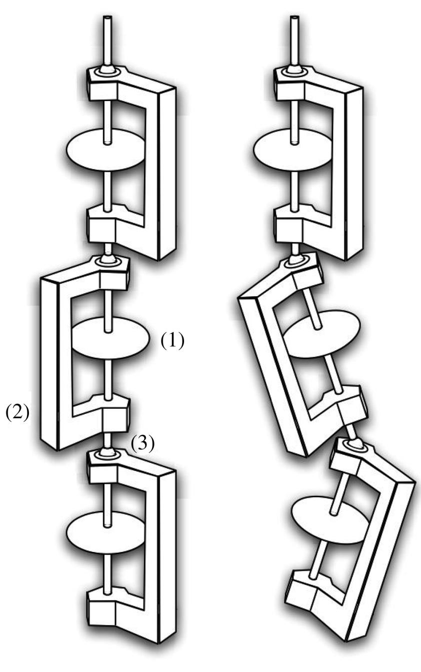


Fig. 3 Sketch of three links of a chain bearing gyroscopes: a) aligned or b) not aligned. The main parts of these links include 1) a rotor, 2) a rigid capsule housing the rotor, and 3) a ball pivot connection with the next link.

To obtain a model of the body that allows us to handle both the macroscopic and microscopic motions at once, we rely on the theory of continua with microstructure, as presented in [9]. The model of the gyrocontinuum obtained through this theory was presented in [5]. It must be noted that this approach leads to a generalization of the results that have previously appeared in the literature: the latter will be recovered under special circumstances. Especially, the assumption that the structure behaves linearly and elastically is not necessary here, which allow us to treat a case such as that of chains.

In [5], the gyroscopes were supposed to interact significantly with the macroscopic motion of the body via the torques generated by changes of moment of momentum imposed upon them by the macroscopic motion. To be more precise, each gyroscope was supposed to be pin-fixed (possibly through pliant or controllable gimbals) to a capsule, in such a way that their centers of gravity always coincide.

The capsule was supposed to be a small part of the continuum; it was supposed to be fully entrained by the macroscopic motion and, being small, it was also supposed to undergo uniform deformations only. The gyroscope was, on the contrary, supposed to be only partly entrained by the capsule, depending on the linkage between them, and, in any case, animated by a spin about its axis that was assigned as relative to the capsule.

To describe the motion of entrainment, it was then necessary to have a precise description of how the linkage between capsule and gyroscope gimbals is effected. If, in the reference configuration of a capsule, a pin-to-pin direction $\mathbf{g}_*$, a particular plane through $\mathbf{g}_*$, of which the normal is the unit vector $\mathbf{a}_*$, and the normal to $\mathbf{g}_*$ within this plane are fixed, a macroscopic motion of gradient $\mathbf{F}$ of that capsule modifies them as

$$\hat{\mathbf{g}} := \frac{\mathbf{F}\mathbf{g}_*}{|\mathbf{F}\mathbf{g}_*|}, \qquad \hat{\mathbf{a}} := \frac{\mathbf{F}^{-T}\mathbf{a}_*}{|\mathbf{F}^{-T}\mathbf{a}_*|}, \qquad \hat{\mathbf{a}} \times \hat{\mathbf{g}} \tag{1}$$

where the $-T$ exponent denotes the inverse of the transpose of $\mathbf{F}$, $\times$ the vector product, and $|\mathbf{v}|$ the norm of $\mathbf{v}$. The entrainment rotation of the capsule $\hat{\mathbf{Q}}$, that is, the orthogonal tensor which moves $\mathbf{g}_*$ into $\hat{\mathbf{g}}$ and the plane normal to $\mathbf{a}_*$ into the plane normal to $\hat{\mathbf{a}}$, is thus completely determined by the macroscopic motion

$$\hat{\mathbf{Q}} = \frac{\mathbf{F}\mathbf{g}_*}{|\mathbf{F}\mathbf{g}_*|} \otimes \mathbf{g}_* + \frac{\mathbf{F}^{-T}\mathbf{a}_*}{|\mathbf{F}^{-T}\mathbf{a}_*|} \otimes \mathbf{a}_* + \left(\frac{\mathbf{F}^{-T}\mathbf{a}_*}{|\mathbf{F}^{-T}\mathbf{a}_*|} \times \frac{\mathbf{F}\mathbf{g}_*}{|\mathbf{F}\mathbf{g}_*|}\right) \otimes (\mathbf{a}_* \times \mathbf{g}_*) \tag{2}$$

where $\otimes$ denotes the tensorial product $(\mathbf{v} \otimes \mathbf{w})_{ij} = v_i w_j$. The total rotation of the gyroscope is given by $\mathbf{Q} = \mathbf{G}\hat{\mathbf{Q}}$, where $\mathbf{G}$ is the relative rotation between gyroscope and capsule, which can be seen as a control variable (see Fig. 4).

Three cases were envisaged: 1) totally unconstrained and possibly controlled relative rotations: $\mathbf{G}$ can be entirely modified by an operator (in any case, it will be assumed that the operator receives no sensible retroaction in the process); 2) null relative precession and variable, possibly controlled speed of relative rotation about the pin-to-pin direction (constraint 1): only a part of $\mathbf{G}$ can be acted upon, as the axis of rotation is fixed to the capsule; and 3) null relative precession and fixed speed of relative rotation about the pin-to-pin direction (constraint 2): as in constraint 1, with a control limited to the sense of rotation.

In any case, the inertia torque per unit mass in the actual configuration is

$$\mathbf{b}_{Ig} := (\mathbf{J}\mathbf{w})^{\cdot} \tag{3}$$

where $\mathbf{J}$ is the inertia tensor of the gyroscope per unit mass in the same configuration ($\alpha_1, \alpha_2 \in \mathbb{R}^+$)

$$\mathbf{J} := \alpha_1 \mathbf{g} \otimes \mathbf{g} + \alpha_2(\mathbf{I} - \mathbf{g} \otimes \mathbf{g}) \tag{4}$$

and $\mathbf{w}$ is the total spin ($\mathbb{E}$ is the Ricci alternating third-order tensor: $E_{ijk} = \pm 1$ if the indices are, respectively, a right or left permutation of 123 and null if neither case occurs):

$$\mathbf{w} = \mathbf{w}^r + \mathbf{G}\mathbf{w}^e, \qquad \mathbf{w}^r := -\tfrac{1}{2}\mathbb{E}\dot{\mathbf{G}}\mathbf{G}^T, \qquad \mathbf{w}^e := -\tfrac{1}{2}\dot{\hat{\mathbf{Q}}}\hat{\mathbf{Q}}^T \tag{5}$$

The relative spin $\mathbf{w}^r$ can be written in terms of relative angular speed ω and relative velocity of precession $\mu\mathbf{p}$, whereas the spin of entrainment $\mathbf{w}^e$ is a function of the macroscopic velocity gradient $\mathbf{L}$; therefore,

$$\mathbf{w} = \omega\mathbf{g} + \mu\mathbf{g} \times \mathbf{p} - \tfrac{1}{2}(\mathbb{E} + \mathbb{H}^T)\mathbf{L} \tag{6}$$

where the third-order tensor $\mathbb{H}$ takes into account the fact that the gyroscopic effect is not only due to the rigid rotations of the capsule bearing the gyroscope (as it can be the consequence of a local rotation of the body), but also due to the deformations of this capsule (if the pin-to-pin direction is not fixed in the deformation process):

$$\begin{aligned}\mathbb{H} := {} & -\hat{\mathbf{g}} \otimes \mathbb{E}\hat{\mathbf{g}} - [(\mathbb{E}\hat{\mathbf{g}}) \otimes \hat{\mathbf{g}}]^T + \hat{\mathbf{a}} \otimes \mathbb{E}\hat{\mathbf{a}} + [(\mathbb{E}\hat{\mathbf{a}}) \otimes \hat{\mathbf{a}}]^T \\ & - 2\,\mathrm{sym}\,(\hat{\mathbf{a}} \otimes \hat{\mathbf{g}}) \otimes (\hat{\mathbf{a}} \times \hat{\mathbf{g}})\end{aligned} \tag{7}$$

Starting from this kinematic description, in [5], the general equations of balance are given for the gyrocontinuum. Here, we focus on a particular case, assuming constraint 1 plus some simplifying assumptions on the nature of internal actions and on the nature of external actions:

1) There is no power of internal actions related to the gradient of the spin of the gyroscopes; in other words, the only internal action acting as a torque on the gyroscopes is equilibrated within each material element.

2) Internal reactions will rise to make constraint 1 effective, and we assume this constraint be *perfect*, with no other internal actions concurring at equilibrium with these reactions.

3) No external torque acts on the axes of the gyroscopes.

The balance of linear momentum in the present configuration is then

$$\mathrm{div}\,\{\mathbf{T} - \tfrac{1}{2}\rho(\mathbb{E} + \mathbb{H})[\mathbf{G}^T(\mathbf{b}_{Ig} + \mathbf{g} \times \mathbf{b}_{Ig} \times \mathbf{g})]\} + \rho\mathbf{b} = \rho\ddot{x} \tag{8}$$

where $\mathbf{T}$ is the (nonnecessarily symmetric) Cauchy stress, $\mathbf{b}$ are the external forces applied to the unit mass, and x is the present position of the center of mass of the material element. The term after $\mathbf{T}$ under the divergence operator represents the contribution to the effective stress of the inertia torque $\mathbf{b}_{Ig}$ given by the gyroscope embedded in the element.

As the microscopic motion can be decomposed into one of rotation about the present pin-to-pin direction and one of change of this direction ([see Eq. (6)], we have the equation of conservation of angular momentum about the present pin-to-pin direction $\mathbf{g}$:

$$-\zeta + \rho\beta = \rho\alpha_1(\omega + \omega^e)^{\cdot} \tag{9}$$

where ζ is the internal equilibrated microtorque about that direction (i.e., an internal torque action per unit volume along $\mathbf{g}$), β is a parallel external torque applied to gyroscopes per unit mass, and ω^e is the angular speed of entrainment, which is possibly very much less than ω.

Finally, the balance of moment of momentum gives the nonsymmetric part of the Cauchy stress:

$$\mathbb{E}\,\mathbf{T} = \zeta\mathbf{g} \tag{10}$$

Notice that the equation of conservation of angular momentum in the plane orthogonal to the pin-to-pin direction has been used to write Eq. (8) and thus need not be considered any further. It showed that the reaction arising in the pins due to constraint 1 is the vector $-\rho\mathbf{b}_{Ig} \times \mathbf{g}$.

Equation (9) allows us to compute the external torque β needed to control the spin of the gyroscopes. For instance, if ζ is due to some kind of internal viscous resistance to relative rotations between rotors and capsule, assuming $\zeta = -\eta\omega$ (η, a viscosity) and $\dot{\omega}^e \ll \dot{\omega}$, we have

$$\frac{\eta}{\rho}\omega + \beta = \alpha_1\dot{\omega} \tag{11}$$

ruling the spin ω of gyroscopes for given β.

Let us take into account constraint 2 and the following hypotheses:

1) The contribution of the deformation of the capsule to the kinetic energy of material elements is negligible.

2) The gyroscopes are all the same and have a privileged spin axis, $\alpha_2 \ll \alpha_1$, so that, from now on, we will drop the index 1 from the first eigenvalue of the inertia tensor of the gyroscopes, $\alpha_1 \to \alpha$, and assume it to be constant.

3) The component of the total spin along the gyroscopic axis is due only to the gyroscopic speed (which will be much larger than the component of the spin of entrainment along that axis).

4) The entrainment spin of the linkage on the capsule will be predominantly given by the vorticity of the macroscopic velocity field.

5) The rate of variation of this vorticity is orthogonal to $\mathbf{g}$ (actually a severe restriction on the kind of deformation processes that can be studied).

Then, it is $\dot{\omega} = 0$ (the external torque $\rho\beta\mathbf{g}$ needs only supply the loss due to ζ), and

$$\mathbf{b}_{Ig} \approx \tfrac{1}{2}\alpha\omega\mathbf{g} \times \mathrm{rot}\,\dot{x} \tag{12}$$

so that $\mathbf{g} \times \mathbf{b}_{Ig} \times \mathbf{g} = \mathbf{b}_{Ig}$. Therefore, the body force due to the torque of inertia of the gyroscopes takes the simplified form suggested in [1,6], and the balance of linear momentum Eq. (8) can be written as follows:

$$\mathrm{div}\,[\mathbf{T} - \tfrac{1}{2}\rho\alpha\omega\mathbb{E}(\mathbf{g} \times \mathrm{rot}\,\dot{x})] + \rho\mathbf{b} = \rho\ddot{x} \tag{13}$$

and in a reference configuration, we have

$$\mathrm{Div}\,[\mathbf{P} - \tfrac{1}{2}\rho_*\alpha\omega\mathbb{E}(\mathbf{g}_* \times \mathrm{Rot}\,\dot{x})] + \rho_*\mathbf{F}^{-T}\mathbf{b} = \rho_*\ddot{x} \tag{14}$$

where $\mathbf{P} = (\det\mathbf{F})\mathbf{F}^{-T}\mathbf{T}\mathbf{F}^{-1}$ is the Piola–Kirchooff stress and capital initials denote differential operators in the reference configuration.

III. Gyroelastic Beams

A first example that illustrates the possibilities of a mechanical system with embedded gyroscopes is that of an Euler–Bernoulli linear elastic beam experiencing small displacements $\mathbf{u}$, an example already partly discussed in [5]. The subject has been widely analyzed using the linear theory [1–4,6,7]; recent studies made some experimental data available [10].

To adapt our general results to this particular case, we need to introduce the beam's geometry, that is, a slender cylinder along x_1, with Navier's kinematic constraint of conservation of normal cross sections; we write the displacement field as follows:

$$\begin{cases} u_1 = \bar{u}_1 - \bar{u}_{3,1}x_3 - \bar{u}_{2,1}x_2 \\ u_2 = \bar{u}_2 \\ u_3 = \bar{u}_3 \end{cases} ; \qquad \bar{\mathbf{u}} = \bar{\mathbf{u}}(x_1, \tau) \tag{15}$$

Developments are straightforward if one takes the axes x_2 and x_3 with origin in the center and along the principal inertia axes of the cross section (calling ϱ_2 and ϱ_3 their relative radii of inertia and E the Young's modulus of the material the beam is made of) and if one neglects the rotational inertia of cross sections in writing the equations of motion of the beam. Furthermore, we take a simplified form for the inertia forces of the gyroscopes, valid if their axes are fixed to the main body and their spin is constant in time and in space (constraint 2).

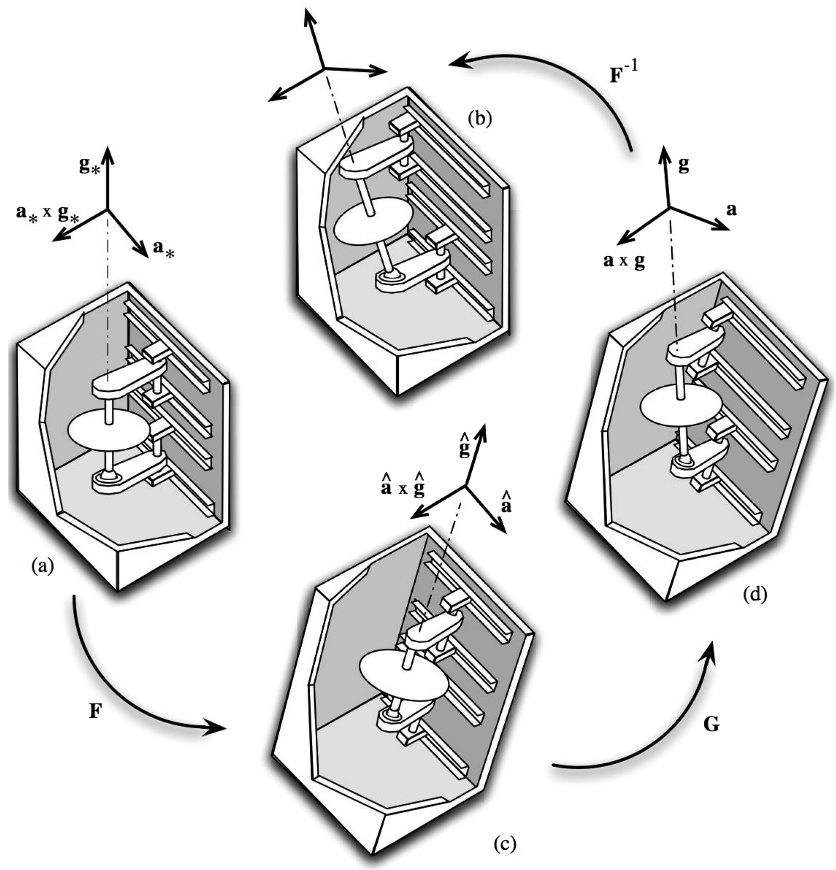


Fig. 4 Kinematics of gyrocontinua: a) reference configuration of the capsule, c) deformed capsule without relative motion of the gyroscope's axis, d) actual configuration of the capsule including deformation and rotation of the gyroscope's axis. Furthermore, b) represents a rotation of the gyroscopic axis in the reference configuration.

Therefore, the two equations of lateral motion of the beam are

$$\begin{cases} \bar{u}_{2,\tau\tau} + \frac{E\varrho_3^2}{\rho}\bar{u}_{2,1111} + \frac{\alpha\omega}{2}\bar{u}_{3,11\tau} = 0 \\ \bar{u}_{3,\tau\tau} + \frac{E\varrho_2^2}{\rho}\bar{u}_{3,1111} - \frac{\alpha\omega}{2}\bar{u}_{2,11\tau} = 0 \end{cases} \tag{16}$$

A. Wave Propagation in Infinite Gyroelastic Beams

Let us study the case

$$\bar{u}_2 = a_2 \exp(\lambda x + \varsigma\tau), \qquad \bar{u}_3 = a_3 \exp(\lambda x + \varsigma\tau) \tag{17}$$

the gyroscopic effect couples the two amplitudes: if $\omega \neq 0$, their ratio is

$$\frac{a_3}{a_2} = \frac{(\alpha\omega/2)\lambda^2\varsigma}{\varsigma^2 + (E\varrho_2^2/\rho)\lambda^4} = -\frac{\varsigma^2 + (E\varrho_3^2/\rho)\lambda^4}{(\alpha\omega/2)\lambda^2\varsigma} \tag{18}$$

and the dispersion equation is

$$\left(\varsigma^2 + \frac{E\varrho_3^2}{\rho}\lambda^4\right)\left(\varsigma^2 + \frac{E\varrho_2^2}{\rho}\lambda^4\right) + \left(\frac{\alpha\omega}{2}\lambda^2\varsigma\right)^2 = 0 \tag{19}$$

leading to two couples of admissible ς for each λ:

$$\varsigma = \pm i\lambda^2\sqrt{\frac{E}{\rho}(\varrho_2^2 + \varrho_3^2) + \left(\frac{\alpha\omega}{2}\right)^2} \times \sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \left(\frac{(E/\rho)\varrho_2\varrho_3}{(E/\rho)(\varrho_2^2 + \varrho_3^2) + (\alpha\omega/2)^2}\right)^2}} \tag{20}$$

and thus 1) if λ is a pure imaginary number, to four waves propagating either back or forth along x_1; 2) if λ is a pure real number, to four (backward or forward) vanishing waves; 3) to the whole set of such waves if λ has both real and imaginary parts. According to Eqs. (17) and (18), each wave corresponds to motion in both x_2 and x_3 directions.

Notice the behavior of the term under the square root in Eq. (20) when ω varies, leading to the two following special cases:

1) The overcontrolled limit case, when the control action exceeds both original flexural stiffnesses of the beam in the sense that $E\varrho_2^2/\rho \ll (\alpha\omega/2)^2$ and $E\varrho_3^2/\rho \ll (\alpha\omega/2)^2$; four of the solutions in Eq. (20) are waves traveling with $\varsigma \approx \pm i(\alpha\omega/2)\lambda^2$, the other four solutions being trivial in the limit. As per Eq. (18), the ratio of amplitudes of the former is then an imaginary number, which corresponds to the propagation of two (backward/forward) screw waves and to two vanishing (backward/forward) such waves.

2) The uncontrolled case $\omega = 0$, giving

$$\varsigma = \pm i\lambda^2\sqrt{\frac{E}{\rho}}\sqrt{\frac{1}{2}(\varrho_2^2 + \varrho_3^2) \pm \frac{1}{2}(\varrho_2^2 - \varrho_3^2)} \tag{21}$$

corresponding to two families of uncoupled waves, obeying the dispersion equations

$$\varsigma_2 = \pm i\lambda^2\varrho_3\sqrt{\frac{E}{\rho}}, \qquad \varsigma_3 = \pm i\lambda^2\varrho_2\sqrt{\frac{E}{\rho}} \tag{22}$$

each family comprising the four classical propagating/vanishing, backward/forward waves.

Any transitions, at a certain place and time, from a solution of the type given in Eq. (20) to one as in Eq. (22), or vice versa, imply the generation of vanishing waves (and the splitting of propagating waves into waves of different characteristics).

Because of the action of gyroscopes, oscillations along the two principal axes of inertia of the beam's cross section are coupled, but an effective stiffness appears that is larger than the natural one and, being a monotonic increasing function of the gyroscopic spin, can be tuned (within limits) acting on that spin [compare Eqs. (20) and (22)]. Such a control will affect the process only through fit changes of the response function to the given external actions, not by the introduction of external control actions. Thus, disturbances of small amplitude can be transmitted at a distance with a velocity that can be tuned, varying in as large a range as is made possible by the control of the spin ω. The property can be used for active insulation of a part of the structure.

B. Harmonic Loading

The stiffening effect of gyroscopes can be observed in the analysis of the response of gyroelastic beams to harmonic loading. Let us consider the kind of solution of Eq. (17) with λ and ς given by the loading

$$\bar{q}_2 = \exp(\lambda x + \varsigma\tau), \qquad \bar{q}_3 = 0 \tag{23}$$

The amplitudes of waves in the two orthogonal planes are coupled by the gyroscopic effect, as already shown in Eq. (18); displacements appear in the x_3 direction due to that effect, but the amplitude in the direction of the loading is smaller than for the uncontrolled beam:

$$a_2 = \frac{1}{\varsigma^2 + ((E\varrho_3^2/\rho) + (\alpha\omega/2)^2\{\varsigma^2/[\varsigma^2 + (E\varrho_2^2/\rho)\lambda^4]\})\lambda^4} \tag{24}$$

C. Free Vibrations of Gyroelastic Beams

The case of beams pinned at the ends $x = 0$ and $x = \bar{x}$ was proposed as an example in [5]; here, we study it under a different angle. No terms with exponential decay exist in this case, due to the boundary conditions, and wavelengths must be an integer fraction of $\bar{x}/\pi$; solutions are then functions of the kind ($j = 2, 3$)

$$\bar{u}_j(x,\tau) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{\bar{x}}\varphi_{jn}(\tau) \tag{25}$$

Especially, for a controlled beam we have

$$\varphi_{jn}(\tau) = a_{jn+}\sin(\sigma_{n+}\tau) + b_{jn+}\cos(\sigma_{n+}\tau) + a_{jn-}\sin(\sigma_{n-}\tau) + b_{jn-}\cos(\sigma_{n-}\tau) \tag{26}$$

where σ_{n+} and σ_{n-} denote the modulus of ς in Eq. (20), when the $\pm$ sign under the square root is taken as $+$ or $-$, respectively, and $\lambda = \mathrm{i}n\pi/\bar{x}$, whereas, in the uncontrolled case, we have the usual general uncoupled solutions

$$\varphi_{jn}(\tau) = a_{jn}\sin(\sigma_{jn}\tau) + b_{jn}\cos(\sigma_{jn}\tau) \tag{27}$$

where σ_{jn} is the modulus of ς_j in Eq. (22) for $\lambda = \mathrm{i}n\pi/\bar{x}$.

The eight coefficients $a_{jn\pm}, b_{jn\pm}$ ($j = 2, 3$, $\pm$ denoting either $+$ or $-$) appearing in Eq. (26) for each n will be determined from the initial conditions (four equations on $\bar{u}_2$, $\bar{u}_3$, and their velocities) and from the gyroscopic coupling between amplitudes of Eq. (18), made here explicit through the four conditions:

$$b_{2n\pm} = a_{3n\pm}/\vartheta_{n\pm}, \qquad b_{3n\pm} = -\vartheta_{n\pm}a_{2n\pm} \tag{28}$$

where

$$\vartheta_{n\pm} := \frac{(\alpha\omega/2)\nu_\pm}{(E\varrho_2^2/\rho) - \nu_\pm^2} = \frac{(E\varrho_3^2/\rho) - \nu_\pm^2}{(\alpha\omega/2)\nu_\pm} \tag{29}$$

and $\nu_\pm$ denotes either of the radical terms in Eq. (20) according to the $\pm$ sign.

Let us consider, from now on, transitions from the uncontrolled to the controlled case, or vice versa, with ω changing at once on the whole beam. First, take a transition uncontrolled/controlled, denote $\bar{\tau}$ the instant when the transition occurs, and consider, for instance, $\bar{u}_3 = 0$ steadily before it and

$$\varphi_{2n}(\tau) = \sin(\sigma_{2n}\tau), \qquad \tau \le \bar{\tau} \tag{30}$$

To shorten the next formulas, let us dispose of the index n and introduce the notations

$$\begin{aligned} \bar{s}_\pm &:= \sin(\sigma_\pm\bar{\tau}), \qquad \bar{c}_\pm := \cos(\sigma_\pm\bar{\tau}) \\ \bar{s}_2 &:= \sin(\sigma_2\bar{\tau}), \qquad \bar{c}_2 := \cos(\sigma_2\bar{\tau}) \end{aligned} \tag{31}$$

The amplitude of controlled waves is given by the solution of the linear system

$$\begin{bmatrix} \bar{s}_+ & \bar{s}_- & \bar{c}_+/\vartheta_+ & \bar{c}_-/\vartheta_- \\ -\vartheta_+\bar{c}_+ & -\vartheta_-\bar{c}_- & \bar{s}_+ & \bar{s}_- \\ \sigma_+\bar{c}_+ & \sigma_-\bar{c}_- & -\sigma_+\bar{s}_+/\vartheta_+ & -\sigma_-\bar{s}_-/\vartheta_- \\ \sigma_+\vartheta_+\bar{s}_+ & \sigma_-\vartheta_-\bar{s}_- & \sigma_+\bar{c}_+ & \sigma_-\bar{c}_- \end{bmatrix} \begin{bmatrix} a_{2+} \\ a_{2-} \\ a_{3+} \\ a_{3-} \end{bmatrix} = \begin{bmatrix} \bar{s}_2 \\ 0 \\ \sigma_2\bar{c}_2 \\ 0 \end{bmatrix} \tag{32}$$

which is

$$\begin{aligned} a_{2-} &= \frac{\bar{s}_-\bar{s}_2}{1 - (\vartheta_-\sigma_-/\vartheta_+\sigma_+)} + \frac{\sigma_2}{\sigma_-}\frac{\bar{c}_-\bar{c}_2}{1 - (\vartheta_-\sigma_+/\vartheta_+\sigma_-)} \\ a_{3-} &= \vartheta_-\frac{\bar{c}_-\bar{s}_2}{1 - (\vartheta_-\sigma_-/\vartheta_+\sigma_+)} + \vartheta_-\frac{\sigma_2}{\sigma_-}\frac{\bar{s}_-\bar{c}_2}{1 - (\vartheta_-\sigma_+/\vartheta_+\sigma_-)} \end{aligned} \tag{33}$$

(notice that the preceding solution exists under condition $\max(\varrho_2, \varrho_3) > 0$) and similar for a_{j+} ($j = 2, 3$) switching indices $+$ and $-$.

A control problem can then be put forward as follows: choose the gyroscopic spin ω and the time $\bar{\tau}$, in such a way that the amplitudes $a_{j\pm}$ obtained from Eqs. (33) satisfy some given criterion. A possible objective of gyroscopic control can be the diversion of energy from one mode to another, justified by the fact that oscillations can be less of a nuisance and eventually better dissipated on some modes than others. The needed criterion can then be built on the basis of some measure of the energy that has been diverted.

To proceed here with handy calculations and give an example, let us look for a transition from the uncontrolled motion of Eq. (30) to a controlled one, as in Eq. (26), and back to an uncontrolled one that maximizes amplitudes along the x_3 direction. Assume that a first transition uncontrolled/controlled occurs at such a time $\bar{\tau}$ that $\bar{c}_2 = 0$, $\bar{s}_2 = 1$ in Eqs. (33):

$$\bar{\tau} = \frac{(4m-3)\pi}{2\sigma_2} \tag{34}$$

for some positive integer m; from Eqs. (28) and (33), we get for $\tau \ge \bar{\tau}$

$$\begin{aligned} \varphi_2(\tau) &= \frac{1}{1 - (\vartheta_-\sigma_-/\vartheta_+\sigma_+)}\cos[\sigma_-(\tau - \bar{\tau})] \\ &\quad + \frac{1}{1 - (\vartheta_+\sigma_+/\vartheta_-\sigma_-)}\cos[\sigma_+(\tau - \bar{\tau})] \\ \varphi_3(\tau) &= \frac{\vartheta_-}{1 - (\vartheta_-\sigma_-/\vartheta_+\sigma_+)}\sin[\sigma_-(\tau - \bar{\tau})] \\ &\quad + \frac{\vartheta_+}{1 - (\vartheta_+\sigma_+/\vartheta_-\sigma_-)}\sin[\sigma_+(\tau - \bar{\tau})] \end{aligned} \tag{35}$$

Then, when a transition back from the controlled to the uncontrolled state occurs (at once on the whole beam) at time $\bar{\bar{\tau}}$, the subsequent uncoupled oscillations are ($\tau \ge \bar{\bar{\tau}}$)

$$\varphi_2(\tau) = \frac{1}{1-(\vartheta_+\sigma_+/\vartheta_-\sigma_-)}\left[\left(\bar{\bar{c}}_+ - \frac{\vartheta_+\sigma_+}{\vartheta_-\sigma_-}\bar{\bar{c}}_-\right)\cos[\sigma_2(\tau-\bar{\bar{\tau}})] - \frac{\sigma_+}{\sigma_2}\left(\bar{\bar{s}}_+ - \frac{\vartheta_+}{\vartheta_-}\bar{\bar{s}}_-\right)\sin[\sigma_2(\tau-\bar{\bar{\tau}})]\right]$$

$$\varphi_3(\tau) = \frac{\vartheta_+}{1-(\vartheta_+\sigma_+/\vartheta_-\sigma_-)}\left[\left(\bar{\bar{s}}_+ - \frac{\sigma_+}{\sigma_-}\bar{\bar{s}}_-\right)\cos[\sigma_3(\tau-\bar{\bar{\tau}})] - \frac{\sigma_+}{\sigma_3}(\bar{\bar{c}}_+ - \bar{\bar{c}}_-)\sin[\sigma_3(\tau-\bar{\bar{\tau}})]\right] \tag{36}$$

where

$$\bar{\bar{s}}_\pm := \sin[\sigma_\pm(\bar{\bar{\tau}}-\bar{\tau})], \qquad \bar{\bar{c}}_\pm := \cos[\sigma_\pm(\bar{\bar{\tau}}-\bar{\tau})] \tag{37}$$

The ratio of the total energy of motion in the two directions after the two control steps is

$$\varepsilon(\bar{\bar{\tau}}) = \frac{[(\bar{\bar{c}}_+/\vartheta_+\sigma_+) - (\bar{\bar{c}}_-/\vartheta_-\sigma_-)]^2 + (1/\sigma_2^2)[(\bar{\bar{s}}_+/\vartheta_+) - (\bar{\bar{s}}_-/\vartheta_-)]^2}{[(\bar{\bar{s}}_+/\sigma_+) - (\bar{\bar{s}}_-/\sigma_-)]^2 + (1/\sigma_3^2)(\bar{\bar{c}}_+ - \bar{\bar{c}}_-)^2} \tag{38}$$

In the limit of an overcontrolled system, φ_3 in Eq. (35) tends to $(\varrho_2/\varrho_3)\sin[\sigma_-(\tau-\bar{\tau})]$ and φ_2 to $\cos[\sigma_-(\tau-\bar{\tau})]$; σ_+ tends to $\alpha\omega/2$ and σ_- to zero; the energy ratio in Eq. (38) tends to infinity as σ_+^2/σ_2^2 and

$$\left(\frac{\sigma_2}{\sigma_+}\right)^2\varepsilon(\tau) \to \frac{(\varrho_2/\varrho_3)^2}{\{1-\cos[(\alpha\omega/2)(\tau-\bar{\tau})]\}^2} \tag{39}$$

Even though such a limit solution has little or no practical import (the controlled frequency tending to infinite, the system actually freezes to the position it takes when the control is switched on and restarts from this position as the latter turns off), it introduces an interesting strategy. Engineering applications can be sought where only the lowest frequencies need be controlled (passive damping eventually becoming sensible above some frequency); then, energy need be diverted from one bending plane to the other only for the components of motion in Eq. (35) at frequency σ_-. In such a case, Eq. (39) shows that a system that tends to be overcontrolled works proficiently to divert energy from the softer to the stiffest bending plane, and should require a transition controlled/uncontrolled at a time

$$\bar{\tau} + \frac{(2k-1)\pi}{\alpha\omega} \tag{40}$$

for some positive integer k.

For moderately controlled systems, let us consider the following problem: find a back transition time (from the controlled to an uncontrolled solution) and a gyroscopic spin during the controlling period, such that both φ_2 and $\dot{\varphi}_2$ are null on the transition, that is, solve the system of equations for the unknown $\bar{\bar{\tau}}$ and ω:

$$\begin{cases} \sigma_- \tan[\sigma_-(\bar{\bar{\tau}}-\bar{\tau})] = \sigma_+ \tan[\sigma_+(\bar{\bar{\tau}}-\bar{\tau})] \\ \frac{\cos[\sigma_-(\bar{\bar{\tau}}-\bar{\tau})]}{\sigma_2^2-\sigma_-^2} = \frac{\cos[\sigma_+(\bar{\bar{\tau}}-\bar{\tau})]}{\sigma_2^2-\sigma_+^2} \end{cases} \tag{41}$$

As multiple solutions might be possible, optimality can be pursued, requiring a minimal duration of the control period, a minimal energetic input, or several other criteria.

In Fig. 5, we represent in the displacement/velocity plane the trajectories of one such solution in both bending directions. The case at issue is obtained considering $\varrho_2 = 2\varrho_3$, $\lambda = 1$, and the shown solution is $(\alpha\omega/2)^2 = 9.3023E\varrho_3^2/\rho$ and $\bar{\bar{\tau}} = 5.4115\bar{\tau}$. A solution for the same system that is optimal in the sense of minimal intervention time is plotted in Fig. 6; it corresponds to $(\alpha\omega/2)^2 = 1.1337E\varrho_3^2/\rho$ and $\bar{\bar{\tau}} = 2.4766\bar{\tau}$.

Finally, in Fig. 7, we show a solution that suggests the behavior of the system if overcontrolled, with $(\alpha\omega/2)^2 = 99.5223E\varrho_3^2/\rho$ and $\bar{\bar{\tau}} = 15.2415\bar{\tau}$: in the limit, the beam freezes in the position where the first control transition is imposed and starts again from this position when the control action is removed.

A use of the previously described theoretical solutions in practice can be contrived to reduce the energy of unwanted oscillations. The energy necessary for the control action is not directly related to the amount of energy to be diverted. Thus, at least in principle, the solutions can help in conceiving energetic efficient devices.

D. Screw Waves Vibrations of Gyroelastic Beams

To conclude this short analysis of gyroelastic beams, notice that screw waves

$$u_2 = \cos\left(\frac{n\pi x}{\bar{x}} \pm \sigma\tau\right), \qquad u_3 = \sin\left(\frac{n\pi x}{\bar{x}} \pm \sigma\tau\right) \tag{42}$$

are also possible solutions of Eq. (16), within the declared assumptions on the geometry of the beam and taking $\varrho_2 = \varrho_3 = \bar{\varrho}$, with dispersion curves (see Fig. 8)

$$\sigma = \pm\left(\frac{n\pi}{\bar{x}}\right)^2\frac{\alpha\omega}{4}\left(1 \pm \sqrt{1+\left(\frac{4}{\alpha\omega}\right)^2\frac{E\bar{\varrho}^2}{\rho}}\right) \tag{43}$$

Notice that, for any choice of $\omega > 0$, two screw waves may travel forward along the beam and two backward; let us denote them through the indices A and B:

$$\frac{n_A^2}{|\sigma_A|}\frac{|\sigma_B|}{n_B^2} = \frac{-1+\sqrt{1+(E\bar{\varrho}^2/\rho)(4/\alpha\omega)^2}}{1+\sqrt{1+(E\bar{\varrho}^2/\rho)(4/\alpha\omega)^2}} \tag{44}$$

The wavelength ratio of waves running in the same direction at equal speed and the speed ratio for equal length can be obtained from the previous formula. This result is also interesting because it has a corresponding case in the linear vibrations of chains, which is the subject of the next section.

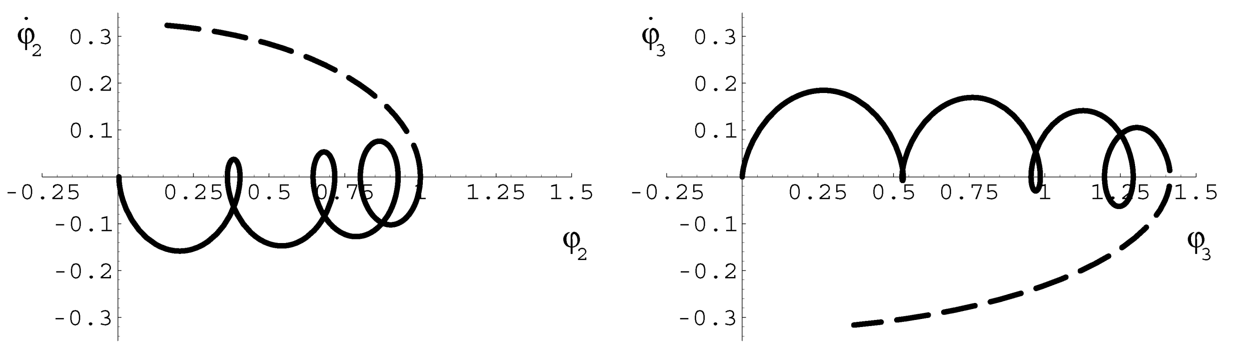


Fig. 5 Trajectories of a controlled beam: φ_2 in the phase diagram on the left and φ_3 on the right. Dashed lines represent the uncontrolled branches of the trajectories, before and after the intervention of the gyroscopes; solid lines represent the controlled branches during such intervention. The beam is at rest in the x_1–x_3 plane before the control $[(\varphi_3, \dot{\varphi}_3) = (0, 0)]$ and in the x_1–x_2 plane after $[(\varphi_2, \dot{\varphi}_2) = (0, 0)]$.

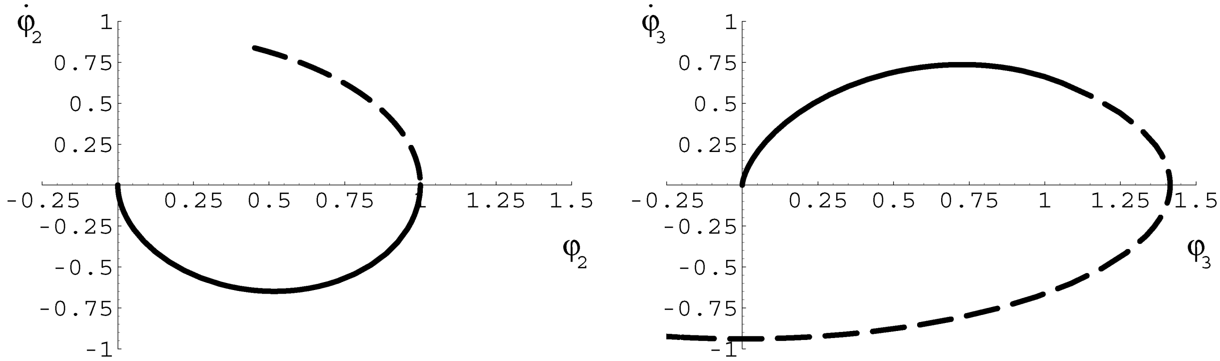


Fig. 6 Trajectories like Fig. 5, but for the minimal time control.

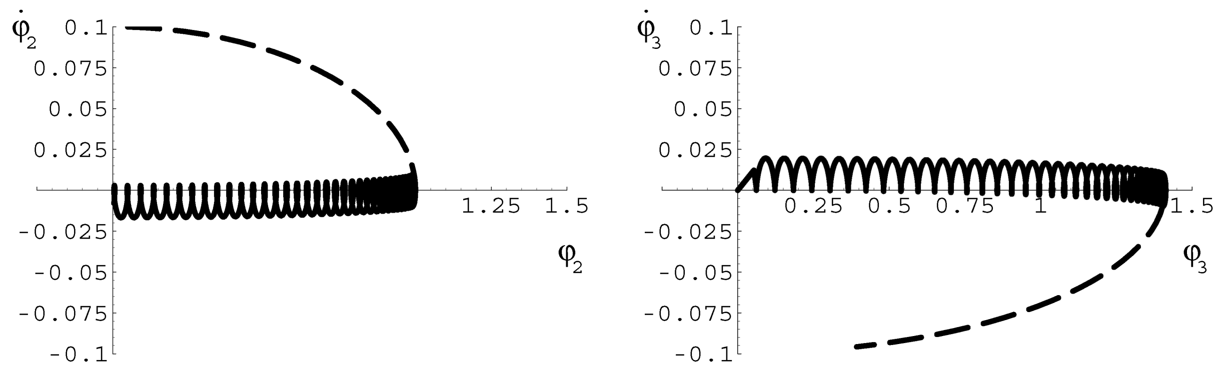


Fig. 7 Trajectories like Fig. 5, but for a very large control action.

IV. Gyrochains

The second example we deal with is that of a chain, or the mathematical model of structural elements like ropes or cables, with embedded gyroscopes.

As suggested in [11], we obtain the field equations of the chain with embedded gyroscopes by integration of the equations of the three-dimensional gyrocontinuum under special assumptions. This procedure can be understood as a constrained search for solutions of the three-dimensional problem that belong to a special class, characterized by some kinematic constraint. In the present case, this constraint acts on the motion of cross sections (sections orthogonal to the mean representative fiber of the chain); the whole section is assumed to move as its center of mass.

Let us consider a segment of unit length, say, parallel to $\mathbf{e}_1$ as a reference configuration $\mathcal{S}_*$ of the mean fiber of the chain; take a coordinate ξ_* along this segment. A motion of the chain is a map, answering to the usual regularity requirements, from $\mathcal{S}_*$ to the present configuration $\mathcal{S}$; call ξ a curvilinear coordinate taken along $\mathcal{S}$.

To describe the motion of the chain as a three-dimensional body, call $x_* \in \mathcal{B}_*$ and $x \in \mathcal{B}$, respectively, its reference and present placements. The two representations are partly connected as $x_{*1} \equiv \xi_*$. For all $\xi_* \in \mathcal{S}_*$, the cross section $\mathcal{A}_*(\xi_*)$ is the set of body particles $(\xi_*, x_{*2}, x_{*3}) \in \mathcal{B}_*$.

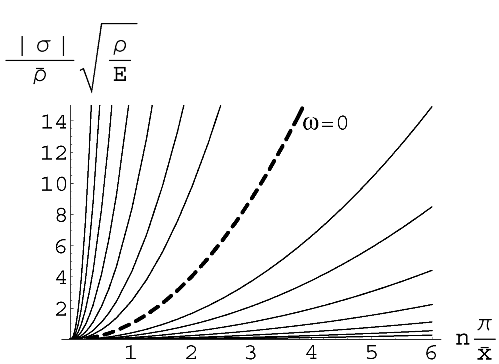


Fig. 8 Plot of $(|\sigma|/\bar{\varrho})\sqrt{\rho/E}$ vs $n(\pi/\bar{x})$ for $(\alpha/4\bar{\varrho})\sqrt{\rho/E}\omega \in \{0, 2^0, \ldots, 2^6\}$. The bold dashed line corresponds to the case $\omega = 0$; for $\omega \to \infty$, one has the two limit solutions $\sigma = 0\ \forall\ n$ and $n = 0\ \forall\ \sigma$.

Assume that there are neither external forces nor couples acting on the boundary of the cross sections and that the mass of each cross section is separately conserved. Actually, to simplify matters, take the mass density to be uniform in the reference configuration. Integration of Eq. (14) on the area $\mathcal{A}_*(\xi_*)$ suggests the introduction of the following functions of ξ_* and time: tension $\mathbf{n}$, reference mass per unit length $\tilde{\rho}_*$ (a constant by assumption), force per unit mass $\tilde{\mathbf{b}}$, and position of the center of mass of cross sections $\tilde{x}$:

$$\begin{aligned} \mathbf{n} &:= \int_{\mathcal{A}_{*1}} \mathbf{P}\mathbf{e}_1\, \mathrm{d}(\text{area})_*, \qquad \tilde{\rho}_* := \int_{\mathcal{A}_*} \rho_*\, \mathrm{d}(\text{area})_*, \\ \tilde{\mathbf{b}} &:= \frac{1}{\tilde{\rho}_*}\int_{\mathcal{A}_*} \rho_* \mathbf{F}^{-T}\mathbf{b}\, \mathrm{d}(\text{area})_*, \qquad \tilde{x} := \frac{1}{\tilde{\rho}_*}\int_{\mathcal{A}_*} \rho_* x\, \mathrm{d}(\text{area})_* \end{aligned} \tag{45}$$

A definition similar to the latter can be given for the reference position of the center of mass $\tilde{x}_*$, which, again to simplify matters, will be assumed to coincide with the point $(x_{*1}, 0, 0)$, and to write the balance of linear momentum (partial derivative with respect to ξ_* is denoted by a derivative index 1):

$$\mathbf{n}_{,1} + \tilde{\rho}_*\tilde{\mathbf{b}} = \tilde{\rho}_*\ddot{\tilde{x}} + \frac{1}{2}\int_{\mathcal{A}_*} \rho_*\alpha\, \mathrm{Div}\,[\omega\mathbb{E}(\mathbf{g}_* \times \mathrm{Rot}\,\dot{x})]\, \mathrm{d}(\text{area})_* \tag{46}$$

The axes of rotation of the gyroscopes are assumed to lay along the link-to-link directions in the chain. These directions are given by the unit tangent vector to the curve (pointing in the direction of increasing curvilinear coordinates):

$$\mathbf{g}_* = \frac{\mathrm{d}\tilde{x}_*}{\mathrm{d}\xi_*} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{g}} = \mathbf{g} = \frac{\partial \tilde{x}}{\partial \xi} \tag{47}$$

Assume also that the chain is at equilibrium in the reference configuration and consider only small displacements from this

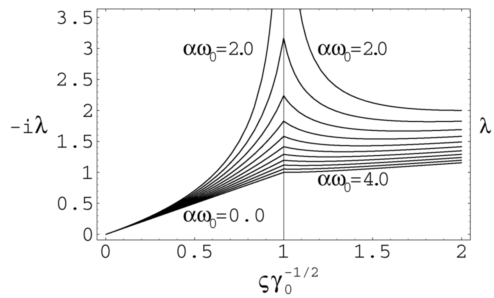


Fig. 9 Dispersion curves for the controlled heavy chain; on the right-hand side of the line $\varsigma/\sqrt{\gamma_0} = 1$, we plot real wavelengths (i.e., propagating waves) for $\frac{1}{2}\alpha\omega_0 \in \{1.0, 1.1, 1.2, \ldots, 1.9, 2.0\}$; imaginary wavelengths (i.e., vanishing waves) are plotted for $\frac{1}{2}\alpha\omega_0 \in \{0.0, 0.1, 0.2, \ldots, 0.9, 1.0\}$ on the left-hand side of the same line.

configuration $\mathbf{u} = \tilde{x} - \tilde{x}_*$; let $\mathbf{e}_1$ be the vertical axis (pointing upward), γ be the acceleration of gravity, and $\tilde{\mathbf{b}} = -\gamma\mathbf{e}_1$. The simplified expression of the force due to the inertia torques of the gyroscopes becomes

$$-\frac{1}{2}\alpha\,\mathrm{rot}\,(\omega\mathbf{g} \times \mathrm{rot}\,\dot{x}) = \frac{1}{2}\alpha\begin{bmatrix} 0 \\ -(\omega\dot{x}_{3,1})_{,1} \\ (\omega\dot{x}_{2,1})_{,1} \end{bmatrix} \tag{48}$$

The first scalar equation of Eq. (46) can be integrated to give $n_1 = \tilde{\rho}_*(\gamma_0 + \gamma x_1)$ ($\tilde{\rho}_*\gamma_0$ a given load at the lower extremity of the chain); taking then

$$\mathbf{n} = n_1\begin{bmatrix} 1 \\ u_{2,1} \\ u_{3,1} \end{bmatrix} \tag{49}$$

the remaining equations can be written (a derivative index τ denotes partial time derivatives):

$$\begin{cases} u_{2,\tau\tau} + [(\gamma_0 + \gamma x_1)u_{2,1}]_{,1} + \frac{1}{2}\alpha(\omega u_{3,1\tau})_{,1} = 0 \\ u_{3,\tau\tau} + [(\gamma_0 + \gamma x_1)u_{3,1}]_{,1} - \frac{1}{2}\alpha(\omega u_{2,1\tau})_{,1} = 0 \end{cases} \tag{50}$$

In these equations, ω must be interpreted as an external control function to be chosen appropriately (within the limits of some given range, depending on the results one wants to seek).

Looking for solutions having the form of screw waves

$$u_2 = \cos(\lambda x_1 - \varsigma\tau), \qquad u_3 = \sin(\lambda x_1 - \varsigma\tau) \tag{51}$$

with λ and ς complex numbers, we find the equation of the control which enables the screw wave to propagate with speed ς along the chain:

$$\omega = \frac{2\gamma}{\alpha\varsigma}x_1 + \omega_0 \tag{52}$$

(ω is a function of x_1, with ω_0 an arbitrary real constant) and the dispersion curves (see Fig. 9)

$$\lambda = \pm\mathrm{i}\frac{\varsigma}{\sqrt{\gamma_0 - \frac{1}{2}\alpha\varsigma\omega_0}} \tag{53}$$

Notice the limit case $\omega_0 = 2\gamma_0/\alpha\varsigma$: waves of infinite length propagating at finite speed. Oscillations result for larger ω_0 if $\varsigma \in \mathbb{R}$; waves attenuate or get amplified for smaller ω_0.

The existence of waves propagating at finite speed through the chain entails the possibility of convecting energy by means of them. A disturbance of small amplitude can then be displaced along the chain by means of such waves, as it can also be done in the studied beams (see Sec. III.D). Comparing Eqs. (43) and (53), it appears in fact that the gyrochain bears the same kind of solution of a fictitious gyroelastic beam. Especially for gyroscopes rotating at very high speeds, the two dispersion equations tend to a parabola

$$\sigma = \tfrac{1}{4}\alpha\omega\lambda^2 \tag{54}$$

V. Conclusions

Structures with embedded distributions of gyroscopes may be considered in applications for their property of being influenceable by the possibly controlled gyroscopic effect. The possible use of high-speed–low-mass rotors to achieve a sensible gyroscopic effect can make such structures effective when, as in aerospacial applications, lightness is sought.

Starting from a previously published theory, we focused here on gyrocontinuum systems as chains or beams, set up their governing equations, and studied some particular solutions under the assumption that the gyroscopic spin can be used as a control variable. Thus, we have shown, in particular, how the gyroscopes can make screw waves of small amplitude propagate through both kinds of structures and the way they affect the linear transversal vibrations of beams.

The second result can be adopted to design a vibration control method, putting an intuitive instance of gyroscopic guidance on a firm basis; by means of fit on–off sequences of the gyroscopic action, the amplitude of vibrations in one plane can be controlled (i.e., made as small as one wishes), diverting the elastic energy to vibrations in the orthogonal plane.

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